CLOS NETWORKS:
A CORRECTION OF THE JACOBAEUS RESULT

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Abstract: In 1950, C. Jacobaeus of LM Ericsson established a method for computing internal blocking probabilities for a point-to-point selection within an interconnection network. Jacobaeus result showed that a three stage network would be "nearly" free of internal blocking if the number of matrices in the second stage was equal to the number of inlets of a first stage matrix plus the number of outlets of a third stage matrix minus one. In 1953, C. Clos of Bell Labs showed that for such a number of second stage matrices referred as the "Clos number" the internal blocking probability should indeed be exactly zero under any traffic hypothesis. This meant that the Jacobaeus result, although quite precise, was not perfectly exact. In this paper we bring a correction to the Jacobaeus calculation, leading to a different result. This new result gives an exact zero internal blocking probability for a number of second stage matrices equal to the "Clos result" showing thus its better accuracy.

Keywords: Interconnection Networks, Link systems, Switching Fabrics, Clos Networks

1 Introduction.

Multistage interconnection networks or “link systems” or “Switching Fabrics” are required when the capacity to be switched in a given node exceeds the capabilities of a single switching element. When small PABXs or routers may be built from a single switching processor, larger nodes have to be organized with multiple stages of switching elements, linked together. Such link systems present the inconvenience of internal blocking, which means that there is a probability that no paths may be found available at a given moment between a given inlet and a given outlet. The art of switch design consist in finding link and switching elements configurations that minimize or even cancel this Internal Blocking probability. The calculation of internal blocking probabilities in multistage connecting networks is therefore a fundamental problem in the switching industry for switching fabric design, when non blocking conditions are not feasible or would be too costly. This used to be the case with analogue switches; it is presently the case with very large asynchronous switches like ATM switches or IP routers. In 1950, C. Jacobaeus of LM Ericsson [JAC] established a method for computing internal blocking probabilities in a point-to-point selection within a three-stage interconnection network. In 1955, C. Y. Lee of Bell Laboratories [LEE] proposed a much simpler, although more approximated method, to compute internal blocking probabilities. The Lee approximation had the advantage of being easily extended to any number of stages and of giving a worst-case assessment since it usually overestimates the internal blocking probability. Consequently the Lee method has been very extensively used in the industry for switching fabric design. In the meanwhile, in 1953, C. Clos of Bell Laboratories [CLOS] showed that a three stage interconnection network would be exactly free of internal blocking, under any traffic condition, if the number r_2 of matrices in the second stage was equal to the number e_1 of inlets of a first stage matrix plus the number of outlets s_3 of a third stage matrix minus one.

Normally, the Clos result should have been predicted from the Jacobaeus result since this last result should give a zero internal blocking probability when the Clos number of secondary matrices is used. However the published Jacobaeus result, although giving a nearly zero value for the internal blocking probability, does not predict an exactly zero value showing that this result is still an approximation. We find that this approximation is due to a simplification done in the calculation of the Jacobaeus result. If this simplification is not allowed, a slightly more complex result is obtained, but with the advantage of providing an exactly zero blocking probability in the Clos condition showing that the Clos result may also be derived from the Jacobaeus theory.

For our derivation of this corrected Jacobaeus result, we consider a three stage Clos Network. The first stage is made of r_1 matrices each with e_1 inlets and s_1 outlets. The second stage is made of r_2 matrices each with e_2 inlets and s_2 outlets. The third stage is made of r_3 matrices each with e_3 inlets and s_3 outlets.

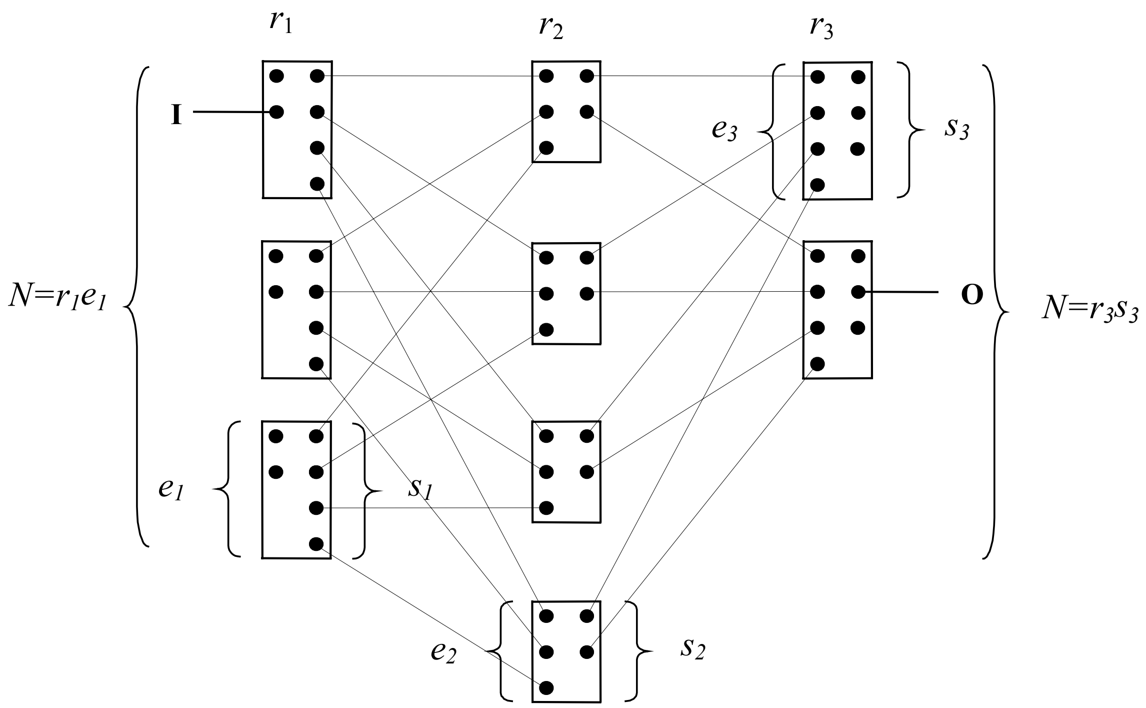

Figure 1: 3 stage point-to-point selection

Such a network is referred as a Clos network when every "r" stage matrix is connected to every "r+1" stage matrix by one and only one link. In this case we have:

$$s_1 = e_3 = r_2$$
$$e_2 = r_1 \ ; \ s_2 = r_3$$

We want to establish a connection between some inlet ***I*** of a first stage matrix and some outlet ***O*** of a third stage matrix. We call $\{x,y\}$ a state of the network where x inlets of this first stage matrix are already busy and where y outlets of the third stage matrix are also busy. We call $B_{x,y}$ the internal blocking probability in state $\{x,y\}$.

In section 2 we will first compute this $B_{x,y}$ probability, as did Jacobaeus. Then, in section 3, we will compute the overall internal blocking probability B_i as the mean value of

$B_{x,y}$ on all possible {*x,y*} states: $B_i = \sum_{x,y} P(x,y) B_{x,y}$, where $P(x,y)$ is the probability of the {*x,y*} state. We will find, when doing so, that our computation of the average differs from the computation of Jacobaeus, leading to a new result that complies exactly with the Clos theory.

2 Internal blocking probability $B_{x,y}$ in the {*x,y*} state.

Theorem 1: the Internal blocking probability $B_{x,y}$ in the {*x,y*} state is :

$$B_{x,y} = 0 \text{ when } x + y < r_2$$

$$\text{and } B_{x,y} = \frac{x!\,y!}{r_2!\left(x + y - r_2\right)!} \text{ when } x + y \geq r_2$$

Proof:

When x inlets of the first stage matrix are busy, x second stage matrices cannot be reached from this first stage matrix. Also, when y outlets of the third stage matrix are busy, only $r_2 - y$ second stage matrices may reach this third stage matrix

Internal blocking occurs when the $r_2 - y$ second stage matrices that can reach our third stage matrix are all belonging to the subset of x second stage matrices that are not reachable from the first stage matrix.

No internal blocking will therefore occur if $r_2 - y > x$ or $x + y < r_2$ so :

$$\text{when } x + y < r_2,\ B_{x,y} = 0$$

When $x + y \geq r_2$, $B_{x,y}$ is given by :

$$B_{x,y} = \frac{\text{number of ways of choosing } \{r_2 \text{ - } y\} \text{ secondary matrices out of the } x \text{ busy one}}{\text{number of ways of choosing } \{r_2 \text{ - } y\} \text{ secondary matrices out of the total } r_2}$$

$$\text{i.e.when } x + y \geq r_2 :\ B_{x,y} = \frac{\mathrm{C}_x^{r_2 - y}}{\mathrm{C}_{r_2}^{r_2 - y}} = \frac{\dfrac{x!}{\left(r_2 - y\right)!\left(x + y - r_2\right)!}}{\dfrac{r_2!}{\left(r_2 - y\right)!\,y!}} = \frac{x!\,y!}{r_2!\left(x + y - r_2\right)!}$$

3 Global internal blocking probability B_i.

We now compute the global internal blocking probability as the mean value of the blocking probabilities in all the {*x,y*} cases:

$$B_i = \sum_{x,y} P(x,y) B_{x,y}$$

If we make a graph of all the possible $\{x, y\}$ states we have to consider that x may vary only from 0 to $\left(e_1 - 1\right)$ and that y may vary only from 0 to $\left(s_3 - 1\right)$. The $\left(e_1 - 1\right)$ and $\left(s_3 - 1\right)$ upper values come from the fact that our ***I*** inlet and ***O*** outlet have to be considered free for establishing a connection between them. This is where our computation differs from the computation of Jacobaeus that was doing the average on all {*x,y*} states with x varying from 0 to e_1 and y varying from 0 to s_3.

Theorem 2: the global internal blocking probability of a three stage Clos network, where each first stage matrices has e_1 inlets , where there are r_2 second stage matrices and where each third stage matrices has s_3 outlets, is given by :

$$\boxed{B_i = e_1!\,s_3!\,\frac{a^{r_2}}{r_2!} \times \frac{\left\lfloor (2-a)^{(e_1+s_3-r_2)} \right\rfloor - \left\lfloor 2 - a^{(e_1+s_3-r_2)} \right\rfloor}{(e_1+s_3-r_2)!}}$$

Proof:

All cases where $x + y < r_2$ give no internal blocking and therefore disappear from the average computation. $\{x, y\}$ states contributing to the average Blocking probability have (x, y) values located within or on the edges of a triangle determined by the lines of equations:

$$y = s_3 - 1$$
$$x = e_1 - 1$$
$$x + y = r_2$$

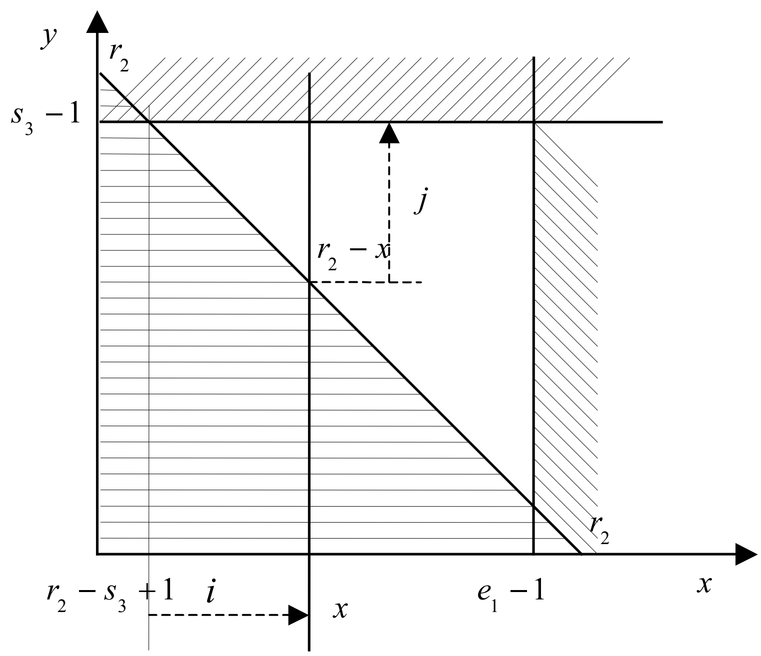

Fig. 2: limit values for average summation

We now consider that the busy states of the *x* inlets of the first stage matrix and of the *y* outlets of the third stage matrix are independent. Although this assumption is not rigorously exact, it is considered as fairly close from reality, mostly when there are a large number of first and third stage matrices and when outgoing trunks of a same trunk group are properly scattered on all the third stage matrices. Measurements justify this assumption. In that case:

$$P(x, y) = P(x)P(y)$$

The internal blocking probability B_i for our point-to-point selection from inlet ***I*** to outlet ***O*** therefore become:

$$B_i = \sum_{(\text{blocking cases})} P(x)P(y)B_{x,y} = \sum_{(\text{blocking cases})} P(x)P(y)\frac{x!\,y!}{r_2!\,(x+y-r_2)!}$$

If we make the blocking cases summation on *x* values first, *x* increases from $r_2 - s_3 + 1$ to $e_1 - 1$ and, for a given value of *x*, *y* increases from $r_2 - x$ to $s_3 - 1$.

$$B_i = \sum_{x=r_2-s_3+1}^{e_1-1} P(x) \sum_{y=r_2-x}^{s_3-1} P(y)\frac{x!\,y!}{r_2!\left(x+y-r_2\right)!}$$

$P(x)$ represents the probability that *x* traffic sources, assumed of equal traffic *a*, among a total of e_1 independent traffic sources are busy. The Bernoulli law describes such a case:

$$P(x) = \mathrm{C}_{e_1}^{x}\; a^x \left(1-a\right)^{e_1-x}$$

$P(y)$ represents the probability that *y* traffic sinks, assumed of equal traffic *a*, among a total of s_3 independent traffic sinks are busy. The Bernoulli law also describes such a case:

$$P(y) = \mathrm{C}_{s_3}^{y}\; a^y \left(1-a\right)^{s_3-y}$$

Assigning these values of *P*(*x*) and *P*(*y*) in the internal blocking expression we get:

$$B_i = \sum_{x=r_2-s_3+1}^{e_1-1} \mathrm{C}_{e_1}^{x}\; a^x\left(1-a\right)^{e_1-x} \sum_{y=r_2-x}^{s_3-1} \mathrm{C}_{s_3}^{y}\; a^y\left(1-a\right)^{s_3-y} \times \frac{x!\,y!}{r_2!\left(x+y-r_2\right)!}$$

$$\text{or } B_i = \frac{e_1!\,s_3!}{r_2!} \sum_{x=r_2-s_3+1}^{e_1-1} \frac{a^x\left(1-a\right)^{e_1-x}}{\left(e_1-x\right)!} \sum_{y=r_2-x}^{s_3-1} \frac{a^y\left(1-a\right)^{s_3-y}}{\left(s_3-y\right)!\left(x+y-r_2\right)!}$$

The internal blocking probability appears as a double summation. By letting $y = r_2 - x + j$, *j* increases from 0 to $\left(s_3-1\right)-\left(r_2-x\right)$ and the summation on *y* becomes a summation on *j*:

$$\sum_y \ldots = \sum_{j=0}^{\left(s_3-1\right)-\left(r_2-x\right)} \frac{a^{\left(r_2-x\right)+j}\left(1-a\right)^{s_3-\left(r_2-x\right)-j}}{\left(s_3-\left(r_2-x\right)-j\right)!\,j!} = a^{r_2-x} \sum_{j=0}^{\left(s_3-1\right)-\left(r_2-x\right)} \frac{a^j\left(1-a\right)^{s_3-\left(r_2-x\right)-j}}{\left(s_3-\left(r_2-x\right)-j\right)!\,j!}$$

By multiplying up and down by $\left(s_3-\left(r_2-x\right)\right)!$, the summation on *y* becomes a binomial sum:

$$\sum_y \ldots = \frac{a^{\left(r_2-x\right)}}{\left(s_3-\left(r_2-x\right)\right)!} \sum_{j=0}^{\left(s_3-\left(r_2-x\right)-1\right)} \mathrm{C}_{s_3-\left(r_2-x\right)}^{j} a^j\left(1-a\right)^{s_3-\left(r_2-x\right)-j} = \frac{a^{\left(r_2-x\right)}}{\left(s_3-\left(r_2-x\right)\right)!}\left(1-a^{s_3-\left(r_2-x\right)}\right)$$

We include this result in the internal blocking probability expression:

$$B_i = \frac{e_1!\,s_3!}{r_2!} \sum_{x=r_2-s_3+1}^{e_1-1} \frac{a^x\left(1-a\right)^{e_1-x}}{\left(e_1-x\right)!} \sum_y \ldots \text{to obtain:}$$

$$B_i = \frac{e_1!\,s_3!}{r_2!} \sum_{x=r_2-s_3+1}^{e_1-1} \frac{a^x\left(1-a\right)^{e_1-x}}{\left(e_1-x\right)!} \times \frac{a^{\left(r_2-x\right)}}{\left(s_3-\left(r_2-x\right)\right)!}\left(1-a^{s_3-\left(r_2-x\right)}\right)$$

$$B_i = e_1! s_3! \frac{a^{r_2}}{r_2!} \sum_{x=r_2-s_3+1}^{e_1-1} \frac{(1-a)^{e_1-x}}{(e_1-x)!(s_3-(r_2-x))!} \times \left(1-a^{s_3-(r_2-x)}\right)$$

By letting $x = r_2 - s_3 + i$ the expression of the internal blocking probability becomes:

$$B_i = e_1! s_3! \frac{a^{r_2}}{r_2!} \sum_{i=1}^{(e_1+s_3-r_2)-1} \frac{(1-a)^{(e_1+s_3-r_2)-i}}{((e_1+s_3-r_2)-i)!\, i!} \times \left(1-a^i\right)$$

$$B_i = e_1! s_3! \frac{a^{r_2}}{r_2!} \frac{1}{(e_1+s_3-r_2)!} \sum_{i=1}^{(e_1+s_3-r_2)-1} \mathrm{C}^i_{(e_1+s_3-r_2)} (1-a)^{(e_1+s_3-r_2)-i} \left(1-a^i\right)$$

$$B_i = e_1! s_3! \frac{a^{r_2}}{r_2!} \frac{1}{(e_1+s_3-r_2)!} \left[\left[\sum_{i=1}^{(e_1+s_3-r_2)-1} \mathrm{C}^i_{(e_1+s_3-r_2)} (1-a)^{(e_1+s_3-r_2)-i}\right] - \left[\sum_{i=1}^{(e_1+s_3-r_2)-1} \mathrm{C}^i_{(e_1+s_3-r_2)} a^i (1-a)^{(e_1+s_3-r_2)-i}\right]\right]$$

The first bracket is equal to: $(2-a)^{(e_1+s_3-r_2)} - 1 - (1-a)^{(e_1+s_3-r_2)}$

The second bracket is equal to: $1 - a^{(e_1+s_3-r_2)} - (1-a)^{(e_1+s_3-r_2)}$

The difference between the two brackets is equal to $(2-a)^{(e_1+s_3-r_2)} + a^{(e_1+s_3-r_2)} - 2$, giving for the internal blocking probability the new formula:

$$B_i = e_1! s_3! \frac{a^{r_2}}{r_2!} \times \frac{\left[(2-a)^{(e_1+s_3-r_2)}\right] - \left[2 - a^{(e_1+s_3-r_2)}\right]}{(e_1+s_3-r_2)!}$$

4 Internal blocking probability B_i and the Clos result.

The classical Jacobaeus result would give a significantly different expression for the internal blocking probability: $B_{i,1} = e_1! s_3! \frac{a^{r_2}}{r_2!} \times \frac{\left[(2-a)^{(e_1+s_3-r_2)}\right]}{(e_1+s_3-r_2)!}$. The Jacobaeus result is too big by the important term $B_{i,2} = e_1! s_3! \frac{a^{r_2}}{r_2!} \times \frac{\left[a^{(e_1+s_3-r_2)} - 2\right]}{(e_1+s_3-r_2)!}$. Many authors have remarked that if the Clos non-blocking value is given to r_2 ($r_2 = e_1 + s_3 - 1$), the Jacobaeus classical formula does not give a result exactly equal to 0. These authors have attributed this peculiar inaccuracy of the classical Jacobaeus result to the independence hypothesis done in the calculation. While we agree that this independence hypothesis does bring some inaccuracy to the method, it should not prevent to predict the Clos result because this result applies under any traffic hypothesis and should therefore be checked even in the case where traffic sources and sinks are independent. We have now seen that the exact reason for the slight inaccuracy of the original formula is the inclusion, in the computation of the mean, of a term that should not be included. If we take the corrected result we get:

$$\text{If}: r_2 = e_1 + s_3 - 1\text{, then } e_1 + s_3 - r_2 = 1 \text{ and } B_i = e_1!\, s_3!\, \frac{a^{r_2}}{r_2!} \times \frac{[(2-a)+a-2]}{(e_1 + s_3 - r_2)!} = 0$$

We therefore find that the corrected Jacobaeus result exactly predicts the Clos result.

5 Conclusions

It is interesting to know that the Jacobaeus method is indeed an exact method for computing internal blocking probabilities under the traffic independence hypothesis, and that the Clos result could have been predicted slightly earlier on this probabilistic approach. However the Clos derivation of this result is a more general derivation since it does not depend on any traffic hypothesis and therefore applies to the traffic independence case. It follows that the independence hypothesis should not prevent the Jacobaeus method to lead to the Clos result. The corrected Jacobaeus computation provides therefore a more accurate mean of computing internal blocking probabilities and is a useful analytical tool.

References

The calculation of internal blocking probabilities in multistage connecting networks is a fundamental problem in switch design when direct implementation of the Clos result [CLOS] is not feasible or would be too costly. This used to be the case with analogue switches; it is presently the case with very large asynchronous switches like ATM switches or IP routers, it may again become the case with pure optical switching. The work of Jacobaeus [JAC] pioneered the study of this problem. A much simpler, although more approximated method has subsequently been proposed by Lee [LEE] and has been very extensively used by the industry. The Lee approximation usually overestimates the internal blocking probability. Variations of the more precise Jacobaeus method are given in the paper by Karnaugh [KAR].

Accounts of these works may be found in the classical textbooks of Misha Schwartz [SCHWA, pp. 544, 552] and of Joseph Y. Hui [HUI, pp. 245, 270]

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